Cyclic correlation based symbol rate estimation

L. Mazet,* Ph. Loubaton

Laboratoire Système de Communication
Université de Marne-la-Vallée
5, boulevard Descartes
Champs sur Marne
77454 Marne-la-Vallée Cedex 2, France
Tel.: (33) 1 60 95 72 90   Fax: (33) 1 60 95 72 14
email: (mazet, loubaton)@univ-mlv.fr

In this paper, we consider the estimation problem of the modulation rate of an unknown emitter using a linear (unknown) modulation. We use the basic observation that the received signal is cyclostationarity, and that the modulation rate is one of its cyclic frequency. The classical estimator [1] based on the maximization in the cyclic frequency domain of a sum of square modulus cyclic correlations sum has poor performance if the excess bandwidth used by the emitter is small. In this paper, we study in detail an estimator introduced by Dandawaté and Giannakis [2], which is based on an appropriate weighted version of the statistics used in [1]. The optimal weighting matrix depends on the unknown parameters, and has to be estimated in practice. However, the optimal matrix depends in general on the unknown cyclic frequencies, and is thus difficult to estimate. The main contribution of this paper is to show that, for low excess bandwidth signals, the optimal weighting matrix actually does not depend on the non zero cyclic frequencies of the received. It can be thus consistently estimated from the cyclic statistics of the observation **at cyclic frequency 0**. We evaluate the performance of the weighted approach, and show that it provide considerable improvement over the traditional estimator of [1].

1 Introduction

Let $x_a(t)$ be the received signal. We assume that it can be written as:

$$x_a(t) = \sum_{n \in \mathbb{Z}} s(n) h_a(t - nT_s) + b_a(t) \tag{1}$$

where, $(s(n))_{n \in \mathbb{Z}}$ is a circular i.i.d symbol sequence, T_s represents the baud rate to be estimated, $h_a(t)$ results from the emission and the reception filters and from the multi-path effects, and is therefore unknown, and $b_a(t)$ is a Gaussian noise with a known variance σ^2. In this paper, we assume as usual that $h_a(t)$ is causal and time limited.

$x_a(t)$ owns the following property: all whole multiple of $\frac{k}{T_s}$ is a cyclic-frequency. However, due to the bandwidth of the usual shaping filters, we will suppose that $x_a(t)$ has only one non-zero positive cyclic frequency, i.e. $\frac{1}{T_s}$. In this paper, we make the reasonable assumption that the signal bandwidth has been roughly estimated. Based on this first evaluation, it is of course possible to sample $x_a(t)$ with a rate greater than $\frac{T_s}{4}$. We denote by $x(k)$ the time series $x(k) = x_a(kT_e)$; as $T_e < T_s/4$, $x(k)$ is cyclostationary, its non zero positive cyclic frequency is $\alpha_0 = \frac{T_s}{T_e}$, and more importantly, the corresponding cyclic spectrum coincides (up to a scaling factor) with the cyclic spectrum of the continuous time signal $x_a(t)$.

2 The basic approach.

For each α, we denote by $\hat{R}_T^{(\alpha)}(\tau)$ the statistics given by

$$\hat{R}_T^{(\alpha)}(\tau) = \frac{1}{T} \sum_{n=0}^{T-1} x(n+\tau)\overline{x(n)} e^{-2\pi i \alpha n} \tag{2}$$

where T is the number of samples. For all τ, when $T \to \infty$, $\hat{R}_T^{(\alpha)}(\tau)$ converges toward 0 when α is different from 0, α_0 and $-\alpha_0$. Let $\mathbf{R}^{(\alpha)} = [R_0^{(\alpha)} \dots R_N^{(\alpha)}]^{\mathrm{T}}$ and $\hat{\mathbf{R}}_T^{(\alpha)} = [\hat{R}_T^{(\alpha)}(0) \dots \hat{R}_T^{(\alpha)}(N)]^{\mathrm{T}}$. A simple estimate of α_0 can be found by solving the

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following maximization problem (see [1]):

$$\max_{\alpha \in \mathcal{I}} \|\hat{\mathbf{R}}_T^{(\alpha)}\|^2 \tag{3}$$

where N is the number of cyclic correlation coefficients taken into account and $\mathcal{I}$ is a search interval included in $]0, 1/2[$. Of course, 0 should not belong to $\mathcal{I}$. The choice of the interval $\mathcal{I}$ has a deep influence on the estimator of α_0 because $\mathcal{R}_T(\alpha) = \mathbf{E}\left[\|\hat{\mathbf{R}}_T^{(\alpha)}\|^2\right]$ may take rather large values around 0. If the excess bandwidth of the received signal is not large enough, $\mathcal{R}_T(\alpha_0)$ may be compared with $\mathcal{R}_T(\alpha)$ for α near 0, and the estimate of α_0 based on (3) may totally fail. To illustrate this, we plot in fig 1 $\mathcal{R}_T(\alpha)$ versus α for $\alpha \neq 0, \alpha_0$. $\mathcal{R}_T(\alpha_0)$ is also represented by a cross. Here, the shaping filter is a square root raised cosine with roll off are set to 0.2 (left) and 0.5 (right). Moreover, $\mathcal{R}_T(\alpha)$ is evaluated by using a classical Gaussian approximation of the vector $\hat{\mathbf{R}}_T^{(\alpha)}$ In order to assess the statistical performance of this standard
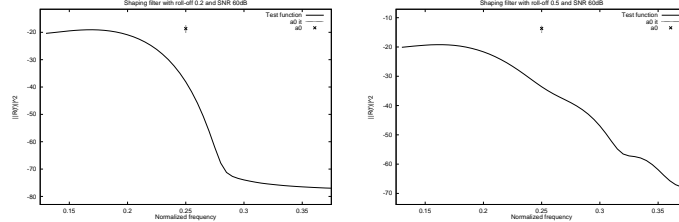

Figure 1: $\mathcal{R}_T(\alpha)$ versus α.

estimate, we have performed some Monte Carlo simulations upon 1000 symbols. In table 1, we give the probability that

Roff-off	0.2	0.5	0.7
Performance (%)	0.00	29.46	99.99

Table 1: Basic estimator performances

the estimate of α_0 lies in the interval $[\alpha_0 - \frac{1}{2T} \ \alpha_0 + \frac{1}{2T}]$ versus the roll off when the searching interval is $[\frac{1}{2}\alpha_0 \ \frac{3}{2}\alpha_0]$. The signal to noise ratio is set to $60dB$ and the number of realizations is equal to 100000. It is quite clear that the performance of the estimate are extremely poor for small roll off.

3 The weighted approach.

The main weakness of the above standard estimate lies on the observation that the mean value of $\mathcal{R}_T(\alpha)$ for $\alpha \neq \alpha_0$ depends on α, and seems to increase when α converges toward 0. In [2], Dandawaté and Giannakis proposed to use the norm of a weighted version of vector $\hat{\mathbf{R}}_T^{(\alpha)}$ in order to estimate (detect) the cyclic frequency. They proposed to use the statistics $\hat{\mathbf{S}}_T^{(\alpha)} = \Gamma(\alpha)^{-\frac{1}{2}}\hat{\mathbf{R}}_T^{(\alpha)}$ where $\Gamma(\alpha)$ is the asymptotic covariance matrix of the estimator $\hat{\mathbf{R}}_T^{(\alpha)}$. By asymptotic covariance matrix, we means that for $\alpha \neq 0, \pm\alpha_0$

$$\sqrt{T}\left(\hat{\mathbf{R}}_T^{(\alpha_0)} - \mathbf{R}^{(\alpha_0)}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \Gamma(\alpha_0)\right)$$

$\hat{\mathbf{R}}_T^{(\alpha)}$ is approximatively a Gaussian centered random vector with covariance matrix $\frac{\Gamma(\alpha)}{T}$ while for $\alpha = \alpha_0$, $\hat{\mathbf{R}}_T^{(\alpha_0)}$ is approximatively a Gaussian random vector with mean $\mathbf{R}(\alpha_0)$ and covariance matrix $\frac{\Gamma(\alpha_0)}{T}$. Using that the symbol sequence is circular, one can show that the vector $\hat{\mathbf{R}}_T^{(\alpha)}$ is asymptotically circular except for $\alpha = 0, \frac{\alpha_0}{2}, \alpha_0$. Therefore, except for those values of α, the asymptotic distribution of the norm square of the statistics

$$\hat{\mathbf{S}}_T^{(\alpha)} = \Gamma(\alpha)^{-\frac{1}{2}}\hat{\mathbf{R}}_T^{(\alpha)} \tag{4}$$

is approximatively a χ^2 distribution with $2N + 2$ degrees of freedom and with variance $\frac{N+1}{T}$. If $\alpha = \alpha_0$, the asymptotic distribution of $\|\hat{\mathbf{S}}_T^{(\alpha)}\|^2$ is Gaussian. Its means and its variance can be calculated in closed form. In this paper, we propose to estimate α by maximizing $\|\hat{\mathbf{S}}_T^{(\alpha)}\|^2$ over an interval $\mathcal{I}$ included in $]0 \ \frac{1}{2}[$. The following figures suggests that the use of $\|\hat{\mathbf{S}}_T^{(\alpha)}\|^2$ may provide better performance than the standard approach described above. In fig, we represent the mean of $\|\hat{\mathbf{S}}_T^{(\alpha)}\|^2$. For $\alpha = \alpha_0$ (which is of course constant) and the mean of $\|\hat{\mathbf{S}}_T^{(\alpha_0)}\|^2$ as well as a 99% confidence interval on the statistics $\|(\hat{\mathbf{S}}_T^{(\alpha_0)}\|^2$. However, the matrix $\Gamma(\alpha)$ is of course unknown. It has therefore to be estimated from the available data. Its closed form expression can be calculated along the lines of [2]. The important to mention is that its general expression depends α_0 and of the cyclic statistics of the observation at this frequency. Therefore, in order to estimate $\Gamma(\alpha)$ consistently, a good initial guess of α_0 has to be obtained. As the standard method of [1] provides very poor estimates,
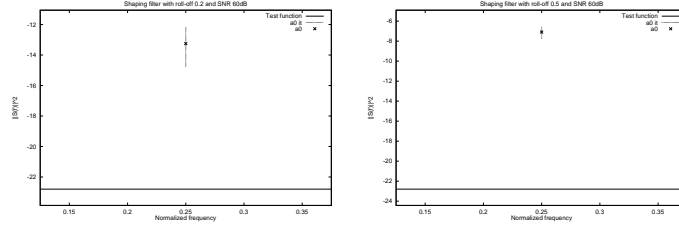
Figure 2: $\mathbf{E}\left[\|\hat{\mathbf{S}}_T^{(\alpha)}\|^2\right]$ versus α

the derivation of such an initial estimator is not obvious. Fortunately, it can shown that, due to the band-limitedness of the shaping filter, the matrix $\Gamma(\alpha)$ does not depend on α_0 for $\alpha > \alpha_0 r$, where r is the roll off. For those values of α, $\Gamma(\alpha)$ is given by

$$[\Gamma(\alpha)]_{(\tau_1,\tau_2)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} S^{(0)}(e^{2i\pi f})\overline{S^{(0)}(e^{2i\pi(f-\alpha)})}e^{2i\pi(\tau_1-\tau_2)f}\mathrm{d}f \tag{5}$$

Hence, it only depends on cyclic statistics at cyclic frequency 0, and can estimated consistently in a straightforward way.

$\Gamma(\alpha)$ is a Toeplitz matrix associated to the "spectral density" $S^{(0)}(e^{2i\pi f})\overline{S^{(0)}(e^{2i\pi(f-\alpha)})}$, which is itself non zero in the interval $[-\frac{1+r}{2}\alpha_0 , \frac{1+r}{2}\alpha_0]$ only. Its numerical rank can be easily approximate by the product between the rows number of $\Gamma(\alpha)$ and the percentage of the used bandwidth which is $(1+r)\alpha_0 - \alpha$. Therefore, $\Gamma(\alpha)$ is very ill conditioned in the absence of noise, and so is its empirical estimate. Therefore in the calculation of the statistic $\Gamma(\alpha)^{-\frac{1}{2}}\hat{\mathbf{R}}_T^{(\alpha)}$ it maybe be useful to replace the inversion by a pseudo-inversion in a dominant sub-space. In order to chose the dimension v, a treader has to be done between the value of $\Gamma(\alpha_0)^{\#\frac{1}{2}}\hat{\mathbf{R}}_T^{(\alpha_0)}$ and the variance of $\Gamma(\alpha)^{\#\frac{1}{2}}\hat{\mathbf{R}}_T^{(\alpha)}$ for $\alpha \neq \alpha_0$ which is equal to $\frac{v}{T}$.

4 Simulations

We have performed Monte Carlo simulations on 100000 experimentations to assess the statistical performance of the weighted estimator.

Roff-off	0.2	0.5	0.7
Performance (%)	99.808	99.674	100.00

Table 2: Weighted estimator performance versus roll-off.

As for table 1, we give, in table 1, the probability that the estimate of α_0 lies in the interval $[\alpha_0 - \frac{1}{2T} \; \alpha_0 + \frac{1}{2T}]$ versus the roll off in the same experimentation context and the dimension of the sub-space in which we pseudo-inverse $\Gamma(\alpha)$ estimate is 2.

It is quiet interesting to show that, even for a roll-off of 0.7 when $\Gamma(\alpha)$ estimate becomes to be biased (the $\Gamma(\alpha)$ estimate we used become to be biased when $\alpha > \alpha_0 r$ and our search interval was $[\frac{1}{2}\alpha_0 \; \frac{3}{2}\alpha_0]$ which is the case for a roll-off of 0.7), the weighted estimator shows very good performances. In fact for such large excess bandwidth the weighting matrix is not so crucial.

5 Conclusion

We present a approach to estimate the symbol rate using the cyclic correlation vector optimally weighted and we assess an estimator of the weighting matrix. Even if the normalized approach is less easy to use than the standard approach of [1], we show that it could be more powerful even the excess bandwidth is very small.

References

[1] W.A. Gardner, *Signal interception, a unifying theorical framework for feature detection*, IEEE trans. on com., Vol. 36 No. 8, August 1988.

[2] A.V. Dandawaté and G.B. Giannakis, *Statistical tests for presence of cyclostationarity*, IEEE trans. on sig. pro., Vol. 42 No. 9, September 1994.